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RESEARCH BULLETIN

ON STOCHASTIC APPROXIMATION

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Abstract

This paper deals with a stochastic process for the approximation of the root of a regression equation. This process was first suggested by Robbins and Monro [1].

The main result here is a necessary and sufficient condition on the iteration coefficients for convergence of the process (convergence with probability one and convergence in the quadratic mean).

ON STOCHASTIC APPROXIMATION

1. Introduction and Summary

In their classical paper Robbins and Monro [1] treated the following problem.

Let $F(y|x)$ be a family of distribution functions depending upon a real parameter x, $-\infty < x < +\infty$, and let $M(x)$,

$$M(x) = \int_{-\infty}^{\infty} y \, dF(y|x) \quad ,$$

be the corresponding regression function. It is assumed that $M(x)$ and $F(y|x)$ are unknown to the experimenter who can, however, take observations on $F(y|x)$ for any value x. Robbins and Monro gave a method for solving stochastically the regression equation

$$M(x) = \alpha \quad , \tag{1}$$

where α is a given number. Under certain conditions on $M(x)$ they were able to construct an iteration procedure $\{X_n\}$ such that X_n converges in probability to the (unique) root θ of (1).

This "Robbins-Monro procedure" is defined as follows. Let $\{a_n\}$ be a fixed sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n = \infty \; , \qquad \sum_{n=1}^{\infty} a_n^2 < \infty \quad . \tag{2}$$

The iteration procedure is then defined recursively as the nonstationary Markov chain $\{X_n\}$ given by

$$X_{n+1} = X_n - a_n(Y_n - \alpha) \; , \quad P(X_1 = a \in R^1) = 1 \quad , \tag{3}$$

where Y_n is a random variable distributed according to $F(y|x = X_n)$ or, in another notation, Y_n is a realization of the random variable $Y(X_n)$.

Later several authors (e.g., Blum [2], Dvoretzky [3]) have shown that even under weaker conditions than those imposed in [1] on $M(x)$, the Robbins-Monro process also converges with probability one and in the quadratic mean.

In this paper we deal with the question of whether it is possible to relax the parameter condition (2). The main result is that the condition

$$a_n \to 0 \quad (n \to \infty) , \quad \sum_{n=1}^{\infty} a_n = \infty \quad , \tag{4}$$

in connection with certain assumptions on $M(x)$, is necessary and sufficient for convergence with probability one and in the quadratic mean. Furthermore, the proof of convergence seems to be more elementary than proofs given by earlier writers.

2. Lemmas

In this section we state and prove two Lemmas which will be needed for the proof of Theorem 1 and Theorem 2 given in section 3.

Lemma A. Let $\{a_i\}$ be a sequence of real numbers. Then

$$\sum_{i=1}^{n} a_i \prod_{j=i+1}^{n} (1 - a_j) = 1 - \prod_{i=1}^{n} (1 - a_i) , \qquad n \geq 1 \quad .^1$$

Lemma B. Let $\{a_i\}$ be a sequence of positive numbers satisfying the condition

$$a_n \to 0 \quad (n \to \infty) , \quad \sum_{n=1}^{\infty} a_n = \infty \quad .$$

Then

$$\sum_{i=1}^{n} a_i^2 \prod_{j=i+1}^{n} (1 - a_j)^2 \to 0 \qquad (n \to \infty) \quad .$$

[1] Throughout this paper the factor of the last term of such a sum equals one.

Lemma A can easily be verified by induction. To prove Lemma B we note first that for any $\varepsilon > 0$ there exists an integer $N_0 = N_0(\varepsilon)$ such that, for all $n \geq N_0$,

$$a_n < \varepsilon , \quad 0 < 1 - a_n < 1 .$$

Hence we get the inequality

$$\sum_{i=1}^{n} a_i^2 \prod_{j=i+1}^{n} (1 - a_j)^2 < \prod_{i=N_0}^{n} (1 - a_i) \left[\sum_{i=1}^{N_0-1} a_i^2 \prod_{j=i+1}^{N_0-1} (1 - a_j)^2 \right] + \varepsilon \sum_{i=N_0}^{n} a_i \prod_{j=i+1}^{n} (1 - a_j) .$$

The factor of ε is less than one by virtue of Lemma A. Because of the divergence of Σa_n there exists for any $M > 0$ an integer $N_1 = N_1(\varepsilon, M)$ such that, for all $n \geq N_1$,

$$\prod_{i=N_0}^{n} (1 - a_i) < \varepsilon \cdot M^{-1} .$$

If we denote

$$\sum_{i=1}^{N_0-1} a_i^2 \prod_{j=i+1}^{N_0-1} (1 - a_j)^2 = M ,$$

it follows immediately that

$$\sum_{i=1}^{n} a_i^2 \prod_{j=i+1}^{n} (1 - a_j)^2 < 2\varepsilon \qquad \text{for all } n \geq N_1 .$$

This completes the proof of Lemma B.

3. Stochastic Approximation of the Root of a Regression Equation

Let us assume that the regression function $M(x)$ corresponding to the family of distribution functions $F(y|x)$, satisfies the following conditions:

(5) $c_1|x - \theta| \leq |M(x) - \alpha| \leq c_2|x - \theta| + c_3 , \quad c_2 \geq c_1 > 0 , \quad c_3 \geq 0$;

(6)
$$\begin{aligned} M(x) &< \alpha & & x < \theta \\ M(x) &= \alpha & \text{for} \quad & x = \theta \\ M(x) &> \alpha & & x > \theta \end{aligned} .$$

The variance of $Y(x)$ is supposed to be uniformly bounded in x ,

(7)
$$\operatorname{Var} Y(x) \leq c_4 < \infty \quad .$$

Then we state the following theorems.

Theorem 1. If conditions (4) through (7) hold, then the stochastic process $\{X_n\}$ given by (3) converges to θ with probability one and in the quadratic mean, $X_n \to \theta$ w.pr. 1 , $E(X_n - \theta)^2 \to 0 \quad (n \to \infty)$.

If we replace condition (5) by

(5')
$$c_1|x - \theta| \leq |M(x) - \alpha| \leq c_2|x - \theta| , \quad c_2 \geq c_1 > 0 ,$$

and if we add the assumption that in a neighborhood of θ $\operatorname{Var} Y(x)$ does not vanish,

(8)
$$\operatorname{Var} Y(x) \geq c_5 > 0 \quad \text{for all} \quad x \in \{x \mid |x - \theta| < \delta , \quad \delta > 0\} ,$$

then the parameter condition (4) is even necessary and sufficient for the convergence of $\{X_n\}$ to θ .

Theorem 2. If conditions (5'), (6), (7), (8) hold, then $\{X_n\}$ converges to θ with probability one and in the quadratic mean if and only if the parameter sequence $\{a_n\}$ fulfills condition (4).

Proof of Theorem 1. We derive a recursion formula for the sequence $E_n = E(X_n - \theta)^2$. From (3) we have

$$(9) \quad E_{n+1} = E(X_{n+1} - \theta)^2 = E[X_n - \theta - a_n(Y_n - \alpha)]^2$$

$$= E_n - 2a_n E[(X_n - \theta)E(Y(x) - \alpha | x = X_n)] + a_n^2 E\{E[(Y(x) - \alpha)^2 | x = X_n]\} .$$

Because of (5) and (6) it follows that

$$E[(X_n - \theta)E(Y(x) - \alpha | x = X_n)] = E[(X_n - \theta)(M(X_n) - \alpha)]$$

$$= E[|X_n - \theta||M(X_n) - \alpha|] \geq c_1 E(X_n - \theta)^2 \geq 0 .$$

From (5) and (7) we get

$$E\{E[(Y(x) - \alpha)^2 | x = X_n]\} = E\{E[(Y(x) - M(x) + M(x) - \alpha)^2 | x = X_n]\}$$

$$= E\{\mathrm{Var}\, Y(X_n) + (M(X_n) - \alpha)^2\} \leq E[c_4 + c_2^2(X_n - \theta)^2 + 2c_2c_3|X_n - \theta| + c_3^2]$$

$$\leq c_4 + 2c_2c_3 + c_3^2 + (c_2^2 + 2c_2c_3)E(X_n - \theta)^2 .$$

Using these inequalities and setting

$$c_4 + 2c_2c_3 + c_3^2 = c_6 , \quad c_2^2 + 2c_2c_3 = c_7^2 ,$$

it follows at once that

$$E_{n+1} \leq (1 - 2c_1a_n + c_7^2a_n^2)E_n + c_6a_n^2 .$$

Because of the convergence of $\{a_n\}$ to zero and $c_7 \geq c_1$ there exists for each constant c_8, $0 < c_8 < c_1$, an integer N_2 such that for all $n \geq N_2$

$$1 - 2c_1a_n + c_7^2a_n^2 < (1 - c_8a_n)^2 .$$

This yields the more convenient inequality

$$E_{n+1} \leq (1 - c_8 a_n)^2 E_n + c_6 a_n^2 , \quad n \geq N_2 \quad .$$

Adding up this inequality from N_2 to n we find

$$(10) \qquad E_{n+1} \leq E_{N_2} \prod_{i=N_2}^{n} (1 - c_8 a_i)^2 + c_6 \sum_{i=N_2}^{n} a_i^2 \prod_{j=i+1}^{n} (1 - c_8 a_j)^2 \quad .$$

The first term of the right-hand side of (10) converges to zero since E_{N_2} is finite and

$$\prod_{i=N_2}^{n} (1 - c_8 a_i) \to 0 \quad (n \to \infty)$$

because of the divergence of Σa_i . Because of Lemma B the same holds for the second term,

$$\sum_{i=N_2}^{n} a_i^2 \prod_{j=i+1}^{n} (1 - c_8 a_j)^2 = (c_8)^{-2} \cdot \sum_{i=N_2}^{n} (c_8 a_i)^2 \prod_{j=i+1}^{n} (1 - c_8 a_j)^2 \to 0 \quad (n \to \infty) \quad .$$

This concludes the proof of convergence in the quadratic mean.

To show that $\{X_n\}$ converges also with probability one we use a method which is similar to that employed by Dvoretzky [3]. We derive the convergence with probability one from the convergence in the mean.

For any pair $\varepsilon > 0, \delta > 0$, there exists an integer $N_3 = N_3(\varepsilon,\delta)$ such that, for all $n \geq N_3$,

$$E_n = E(X_n - \theta)^2 < \varepsilon\delta^2 \quad .$$

We modify the sequence $\{X_n\}$:

(11) $\quad X'_n = X_n \quad$ for all $\quad n \leq N_3$

(12) $$X'_{n+1} = \begin{cases} X'_n - a_n(Y_n - \alpha) & \text{if} \quad |X'_n - \theta| < \delta \\ X'_n & \text{otherwise} \end{cases} \quad , \quad n \geq N_3 \quad .$$

Y_n denotes now a realization of the random variable $Y(x = X'_n)$ instead of $Y(x = X_n)$.

Equations (11) and (12) imply that also

(13) $$E(X'_n - \theta) < \varepsilon\delta^2 \quad \text{for all} \quad n \geq N_3 \quad .$$

If $|X_j - \theta| \geq \delta$ for any $j > N_3$, it follows from (12) that $|X'_n - \theta| \geq \delta$ for all $n \geq j$, and we obtain, for all $n > N_3$,

$$P\{\max_{N_3 < j \leq n} |X_j - \theta| \geq \delta\} \leq P\{|X'_n - \theta| \geq \delta\} \quad .$$

Together with (13) this implies that $\{X_n\}$ converges with probability one to θ , i.e.,

$$P\{\sup_{j > N_3} |X_j - \theta| > \delta\} < \varepsilon \quad .$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Since Theorem 1 implies the sufficiency of parameter condition 4, it remains only to prove that the parameter condition (4) is necessary for the convergence of $\{X_n\}$ to θ . We assume that the sequence $\{E_n\}$ converges to zero even in the case when we use a parameter sequence $\{a_n\}$ which does not satisfy condition (4). We show that this assumption yields a contradiction.

The parameter sequence $\{a_n\}$ under consideration has to fulfill exactly one of the following conditions:

(a) $$\sum_{i=1}^{\infty} a_i < \infty \quad ;$$

(b) there exists a subsequence $\{a_{n_i}\}$ and a constant $L > 0$ such that $a_{n_i} \geq L > 0$ for all i .

From the asserted convergence of E_n to zero it follows--as we have seen--that X_n converges to θ with probability one. Therefore and because of (8) there exists an integer N_4 such that, with probability one,

$$\min_{n \geq N_4} \operatorname{Var} Y(X_n) \geq c_5 > 0 \quad .$$

In the parameter case (a), which implies $a_n \to 0$ $(n \to \infty)$, we can further assume that N_4 is so large that

$$0 < 1 - 2c_2 a_n + c_1^2 a_n^2 < 1 \qquad \text{for all} \quad n \geq N_4 \quad .$$

Hence it follows from (9) by similar arguments to those used before that

$$\begin{aligned} E_{n+1} &\geq E_n - 2c_2 a_n E_n + a_n^2 [E(\operatorname{Var} Y(X_n)) + c_1^2 E_n] \\ &\geq (1 - 2c_2 a_n + c_1^2 a_n^2) E_n + c_5 \cdot a_n^2 \quad , \qquad n \geq N_4 \quad . \end{aligned}$$

Again there exists for each $c_9 > c_2$ an integer $N_5 = N_5(c_3) \geq N_4$ such that

$$E_{n+1} \geq (1 - c_9 a_n)^2 E_n + c_5 a_n^2 \qquad \text{for all} \quad n \geq N_5 \quad .$$

Hence we get for the parameter case (a)

$$E_{n+1} \geq E_{N_5} \cdot \prod_{i=N_5}^{n} (1 - c_9 a_i)^2 + c_5 \sum_{i=N_5}^{n} a_i^2 \prod_{j=i+1}^{n} (1 - c_9 a_j)^2 \geq$$

$$c_5 \cdot f \cdot \sum_{i=N_5}^{n} a_i^2 = c_{10} \quad ,$$

where

$$f = \prod_{j=N_5+1}^{\infty} (1 - c_9 a_j)^2$$

is greater than zero because of the convergence of Σa_i . Hence we have $E_n \geq c_{10} > 0$ for all $n \geq N_5$, which implies the desired contradiction.

In case (b) we get the contradiction immediately by considering the sequence of inequalities

$$E_{n_i} \geq c_5 a_{n_i}^2 \geq c_5 \cdot L > 0 \qquad \text{for all } n_i \geq N_4 \ .$$

This completes the proof of Theorem 2.

4. Concluding Remarks

The crucial assumptions which lead to the weakening of the parameter condition (2) are the two assumptions contained in (5) and (5'), respectively. One of the assumptions in (5),

$$(14) \qquad |M(x) - \alpha| \leq c_2 |x - \theta| + c_3 \quad ,$$

cannot be relaxed as it was pointed out, e.g., by A. Dvoretzky ([3], p. 51). However, it might be interesting to know if the validity of Theorems 1 and 2 is affected by weakening the other assumption made in (5) and (5'),

$$(15) \qquad c_1 |x - \theta| \leq |M(x) - \alpha| \quad .$$

In particular, we may ask if it is possible to replace (15) by the usual condition (e.g., Blum [2], p. 382)

$$\inf_{\delta_1 \leq |x-\theta| \leq \delta_2} |M(x) - \alpha| > 0 \qquad \text{for every pair of numbers } (\delta_1,\delta_2) \text{ with } 0 < \delta_1 < \delta_2 < \infty \quad .$$

In practice, however, condition (14) and (15) will cause no trouble, because in almost all instances the experimenter knows that the root θ lies in some finite interval $[C_*,C^*]$. Therefore he can replace the iteration procedure (3) by the bounded stochastic approximation process

$$X_{n+1} = \begin{cases} C_* & X_n - a_n(Y_n - \alpha) < C_* \\ X_n - a_n(Y_n - \alpha) \quad \text{if} & C_* \leq X_n - a_n(Y_n - \alpha) \leq C^* \\ C^* & X_n - a_n(Y_n - \alpha) > C^* \end{cases} \quad .$$

In this situation (14) and (15) do not seem very restrictive.

References

[1] Robbins, H., and Monro, S. A stochastic approximation method. The Annals of Mathematical Statistics, 1951, 22, 400-407.

[2] Blum, J. Approximation methods which converge with probability one. The Annals of Mathematical Statistics, 1954, 25, 382-386.

[3] Dvoretzky, A. On stochastic approximation. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability I, 39-55. University of California Press, 1956.